1. $\Omega = \Omega_o = \text{const}$

It is the case when the outer sphere suddenly starts spinning with a constant angular speed Ω_o about the z-axis due to some perturbation at time $t = 0$. Then from Eq. (19),

$$\omega = \frac{2\Omega_o}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\beta x(e^{-\nu\beta^2 t}-1) \tag{21a}$$

which can be reduced to

$$\omega = \frac{\Omega_o x}{h}+\frac{2\Omega_o}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\nu\beta^2 t}\sin\beta x \tag{21b}$$

The solution is seen to consist of a steady-state term and a transient which dies out as $t\to\infty$.

Differentiating Eq. (21b) with respect to x and then substituting $x = 0$ in the resulting expression, one obtains

$$\left(\frac{\partial\omega}{\partial x}\right)_{x=0} = \frac{\Omega_o}{h}\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-\nu\beta^2 t}\right], \quad t>0 \tag{22}$$

2. $\Omega = \Omega_o \sin(\alpha t+\varepsilon)$

It is the case when the outer sphere starts performing a harmonic motion about the z-axis with an initial angular speed $\Omega = \Omega_o \sin\varepsilon$ at $t = 0$. The solution, as obtained from Eq. (19), can be expressed as

$$\omega/\Omega_o = A\sin(\alpha t+\varepsilon+\psi)+ 2\pi\nu\sum_{n=1}^{\infty}\frac{n(-1)^n(\nu n^2\pi^2\sin\varepsilon-\alpha h^2\cos\varepsilon)}{\nu^2 n^4\pi^4+\alpha^2 h^4}e^{-\nu\beta^2 t}\sin\beta x \tag{23}$$

where

$$A = \left[\frac{\cosh 2kx-\cos 2kx}{\cosh 2kh-\cos 2kh}\right]^{1/2} \tag{24}$$

$$\psi = \tan^{-1}\left(\frac{\tan kx}{\tanh kx}\right)-\tan^{-1}\left(\frac{\tan kh}{\tanh kh}\right) \tag{25}$$

and

$$k = (\alpha/2\nu)^{1/2} \tag{26}$$

The solution is again seen to consist of a steady-state term and a transient which dies out as $t\to\infty$.

Differentiating Eq. (23) with respect to x and then evaluating the resulting expression in the limit as $x\to 0$, the following is obtained:

$$\frac{1}{\Omega_o}\left(\frac{\partial\omega}{\partial x}\right)_{x=0} = D\sin(\alpha t+\varepsilon+\delta)+ \frac{2\pi^2\nu}{h}\sum_{n=1}^{\infty}\frac{n^2(-1)^n(\nu n^2\pi^2\sin\varepsilon-\alpha h^2\cos\varepsilon)}{\nu^2 n^4\pi^4+\alpha^2 h^4}e^{-\nu\beta^2 t}, \quad t>0 \tag{27}$$

where

$$D = 2k/(\cosh 2kh-\cos 2kh)^{1/2} \tag{28}$$

and

$$\delta = \pi/4-\tan^{-1}(\tan kh/\tanh kh) \tag{29}$$

Now, the viscous torque N at any time $t>0$ may be easily evaluated for the above two cases from Eq. (18) by substituting therein the proper expression for $(\partial\omega/\partial x)_{x=0}$.

III. Conclusions

A time-dependent solution of the viscous flow within the gap between two concentric spheres, when the inner sphere is fixed and the outer suddenly starts spinning about one of its axes, has been obtained under the assumption that the secondary flow is zero and that the gap width is small compared to the radii of the two spheres. Only an incompressible fluid with constant viscosity has been considered. An expression for the viscous torque on the inner sphere, which is also time-dependent, has been provided. The results may be considered the first approximation to the solution for the more general case involving eccentricity, the effect of which may be studied by perturbation techniques.

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Reversed Boundary-Layer Flows with Variable Fluid Properties

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In a comprehensive parametric study it is shown that the characteristics of reversed laminar boundary-layer flows are strongly influenced by the variations of the fluid transport properties. The existing data for constant density-viscosity flows are shown to be not only seriously in error relative to the results for flows with realistic variation of fluid properties but also to exhibit incorrect trends. The implications to corresponding turbulent flow calculations are noted.

Solutions of self-similar, constant density-viscosity product $[C = (\rho\mu)/(\rho\mu)_e = 1.0]$ fluid, laminar boundary-layer flows have been used extensively as the basic data for approximate analyses of flows with separation and reattachment (e.g., Lees and Reeves[1] and Gautier and Ginoux[2]). The use of computed boundary-layer profiles is considered to give improved accuracy over analyses employing simple polynomial representations of profiles. Since this approach is currently under consideration it seems appropriate to examine the relation of constant C data to the corresponding results based on more realistic fluid property variations.

Following the work of Stewartson,[3] Rogers[4] and Keller and Cebeci[5] have performed extensive studies of reversed boundary-layer flows of constant density-viscosity product fluids, with

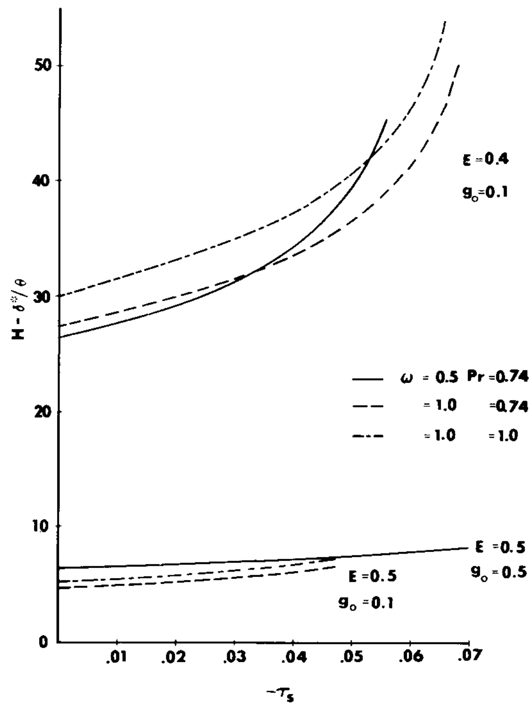

Fig. 1 Variation of the shape factor H with surface shear stress.

Presented at the Open Forum Session of the AIAA 8th Thermophysics Conference, July 16–18, 1973, Palm Springs, Calif.; submitted September 6, 1973.

Index category: Boundary Layers and Convective Heat Transfer—Laminar.

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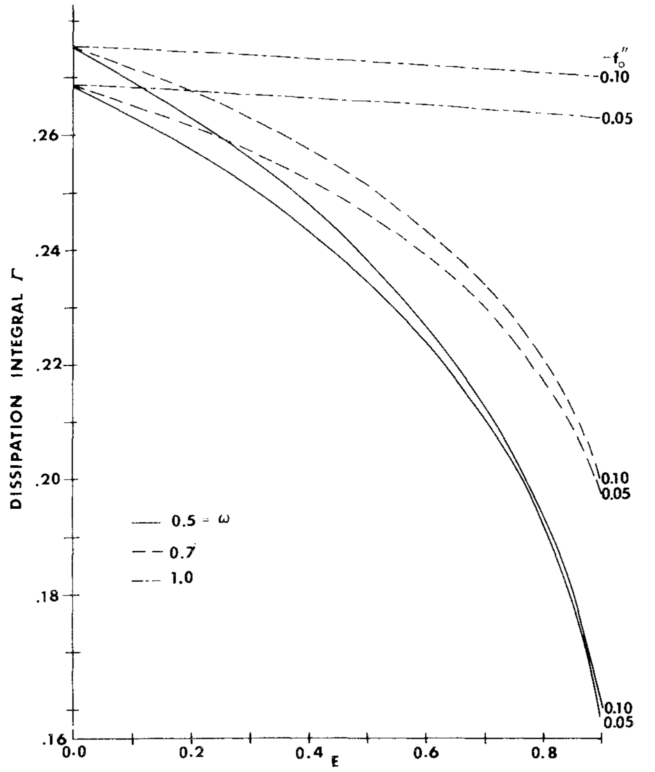

Fig. 2 Variation of the dissipation integral with E at constant f_o''.

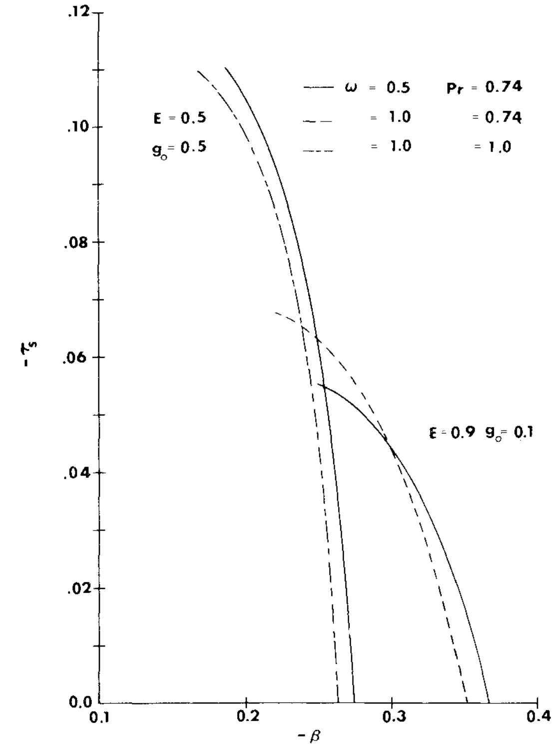

Fig. 4 Correspondence between the pressure gradient and surface shear stress.

nonunity Prandtl numbers in the case of Ref. 4. Using similar, simplified, fluid properties, Libby and Liu[6] found multiple solutions for these problems. It does not appear that the problem warranted that much attention but the results were quite interesting from a mathematical point of view.

Variable fluid properties flows are parameterized most easily when the viscosity $\mu \propto h^{\omega}$, with h being the static enthalpy and ω a constant between 0.5 and 1.0. The density ρ varies as h^{-1} and the Prandtl number Pr is taken to be constant, 0.74 or 1.0. The results reported here were obtained using essentially the same program as was used in Ref. 7 where the accuracy and validity of the procedure were established.

In Fig. 1 the shape factor H = displacement thickness/momentum thickness is shown as a function of the dimensionless wall shear stress τ_s for a range of energy parameters $E = u_e^2/(2H_e)$ and wall to total enthalpy ratios g_o. The dissipation integral $\Gamma = \int_o^\infty (Cf'')f''\,d\eta$ for two specified velocity gradients at the wall is seen from Fig. 2 to be almost independent of E for degenerate fluid properties and strongly dependent on E when realistic fluid properties are employed. The inviscid velocity gradient parameter β shown in Fig. 3 as a function of E is again seen to be a strong function of the fluid property variation.

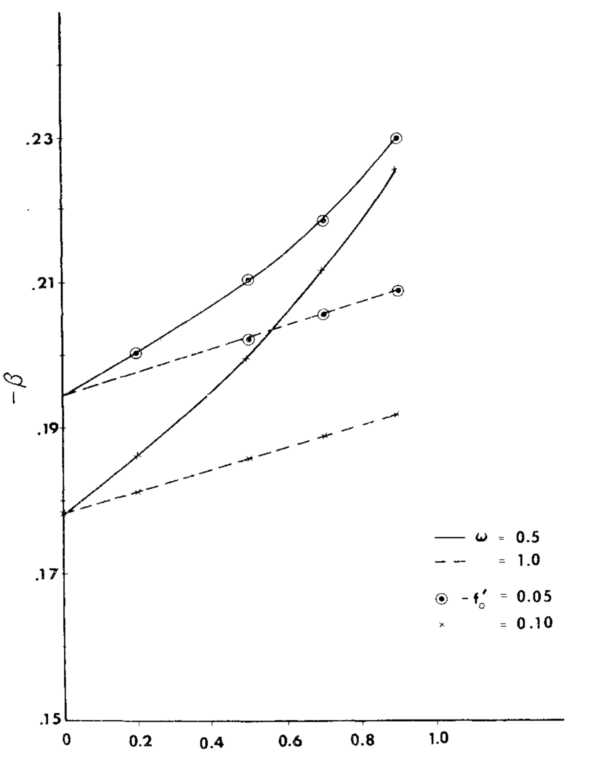

Fig. 3 Variation of the pressure gradient β, at constant f_o'', with E for adiabatic flows.

The results demonstrate that reversed boundary-layer flows are very strongly dependent on the variation of fluid physical properties. Data obtained using degenerate fluid properties bear little relationship to those yielded by calculations employing realistic variations of C. In view of this, the claim that the use of computed boundary-layer profiles in place of polynomial profile approximations improves the accuracy appears to be rather difficult to justify. More significantly, the approach such as that used in Refs. 1 and 2 will be difficult to extend to turbulent flows in which the uncertainties in the variation of eddy viscosity may be considerable.

It might be of some interest to note that, using the present technique, solutions could be obtained for increasing β, starting from the separation value, until an apparent minimum value of surface shear stress was obtained. Solutions could not be continued beyond that point. In contrast with the solution methods of Refs. 3–6 the present technique imposes analytically an exponential decay on f'' and g' in the outer reaches of the boundary layer. This might be a possible explanation for the failure to continue solutions to the origin of the $\tau_s - \beta$ graph shown in Fig. 4. Note should be taken of the extreme sensitivity of τ_s to changes in β near the separation point.

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Wall Shear at a Three-Dimensional Stagnation Point with a Moving Wall

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Introduction

Sears and Telionis[1] have recently treated the question of the proper generalization of Prandtl's criterion for separation in other than the classical case of two-dimensional, steady flows over fixed walls. In their considerations, the solution of Rott[2] for the flow at a two-dimensional stagnation point when the wall is moving, the case of a circular cylinder rotating about its axis normal to an oncoming stream, is used to illustrate the inadequacy of associating zero wall shear with "separation." In this flow there is a line along which the wall shear is zero without the effects usually thought to accompany separation, e.g., a rapid increase in boundary-layer thickness or any other sign of breakdown of flow of the boundary-layer type. In this regard, it is noteworthy that relative to a coordinate system with respect to which flow is steady, the surface streamlines which illuminate separation in the case of fixed walls are not interesting when the wall is moving.

From the point of view of these considerations it appears interesting to treat the more general stagnation point flow over a moving wall. The flow situation corresponds, for example, to the stagnation point on an ellipsoid of revolution rotating about one of its axes normal to the oncoming stream. In the case of three-dimensional separation over fixed surfaces we consider a surface line which is the locus of tangents of surface streamlines from regions upstream and downstream (in a sense normal to the line) of separation.[3] The vector representing the surface shear is tangent to this line. As suggested above, if the wall is moving, the surface streamlines are not interesting, whereas the line along which the wall shear is tangent is significant.

Received September 10, 1973. The author gratefully acknowledges helpful suggestions by W. R. Sears.

Index categories: Fluid Dynamics; Boundary Layers and Convection Heat Transfer—Laminar.

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Table 1 Wall values for several values of the parameter c

c	$f''(0)$	$g''(0)$	$F'(0)$	$G'(0)$
1	1.3119	1.3119	−0.9387	−0.9387
0.5	1.2669	0.9981	−0.8690	−0.7637
0	1.2326	0.5705	−0.8113	−0.5705
−0.5	1.2302	−0.1115	−0.8044	−0.5115

Before proceeding we note that Rott pointed out the possibility of generalizing his solution to the case of the three-dimensional stagnation point but apparently this possibility has not been exploited.† We also note that Davey[4] provides a general analysis of the three-dimensional stagnation point in a fluid which is advancing toward the body and is rotating about the same axis as is the body.

Analysis

Our analysis is an extension of Davey[5] and Libby[6] and our notation is thus standard. The surface is the xy-plane; the external flow is characterized by

$$u_e = \alpha x \qquad v_e = c\alpha y \tag{1}$$

where the coordinate system is selected so that $c \leq 1$. The classic, well-known cases of stagnation point flow, i.e., the axisymmetric and two-dimensional cases, correspond respectively to $c = 1$ and $c = 0$. The surface ($z \equiv 0$) is considered to be moving with velocity components u_w, v_w.

The similarity variable for the general, three-dimensional case is

$$\eta = z(\alpha/\nu)^{1/2} \tag{2}$$

We assume the following form for the velocity components tangent to the surface

$$u = u_w F(\eta) + \alpha x f'(\eta) \qquad v = v_w G(\eta) + c\alpha x g'(\eta) \tag{3}$$

The continuity equation supplemented by the inessential but convenient assumption that $f(0) = g(0) = 0$ leads to the third velocity component

$$w = -(\alpha\nu)^{1/2}(f + cg) \tag{4}$$

Substitution of Eqs. (3) and (4) into the x-wise and y-wise momentum equations and collection of powers of x, i.e., x^0, and x, leads to the equations for $F(\eta)$ and $G(\eta)$

$$F'' + (f + cg)F' - f'F = 0 \qquad G'' + (f + cg)G' - cg'F = 0 \tag{5}$$

and for $f(\eta)$ and $g(\eta)$

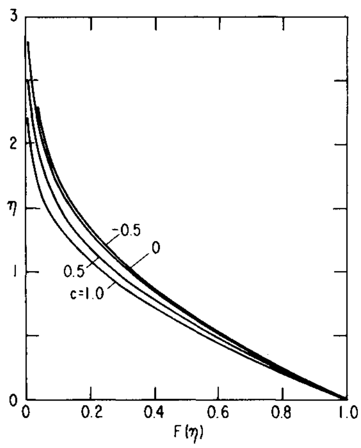

Fig. 1 The normalized x-wise velocity profiles for various values of the parameter c.

† The author is indebted to N. Rott and A. Davey for trying to recall previous publication of the present solutions.